Chapter 5: Series

In this chapter, we define series of complex numbers. Whenever we encounter series of real numbers, we will freely use results that are proved in calculus.

Our main interest is the representation of functions as infinite series. Among other thing, we will prove:

(1) A function f that is analytic on a disk $D_R(z_0)$ has a convergent power series representation on that disk:

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z-z_0)^k .$$

Conversely, every power series $\sum_{k=0}^{\infty} a_n (z-z_0)^k$ is an analytic function on its domain of convergence.

(2) A function that is analytic on an annulus $R_1 < |z-z_0| < R_2$ has a convergent series representation on that annulus:

$$f(z) = \sum_{k=0}^{\infty} a_k (z-z_0)^k + \sum_{k=1}^{\infty} b_k \frac{1}{(z-z_0)^k}$$

with coefficients

$$a_k = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{k+1}}\, dz \quad \text{and} \quad b_k = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{-k+1}}\, dz$$

where C is any pos. oriented simple closed contour in the annulus and surrounding z_0.

In fact, (2) provides a method for computing integrals over contours that surround a singular point! Suppose that f has a singularity at z_0, but is analytic everwhere else in a deleted neighborhood $D_R(z_0)\setminus\{z_0\}$ of z_0. Then f is analytic on the annulus $0<|z-z_0|<R$. If C is any simple closed positively oriented contour around z_0 and lying inside the annulus, then according to (2),

$$2\pi i\, b_1 = \int_C f(z)\, dz.$$

In other words, we can compute the contour integral of f about a singularity just by computing the coefficient b_1 in the series! This is the beginning of the Theory of Residues.

Sequences

Definition (Sequences) A sequence of complex numbers is a complex-valued function z whose domain is the set of positive integers $\mathbb{N}$. We write $z_n = z(n)$ for the value of z at $n \in \mathbb{N}$. We think of the values as occurring in a certain order:

$$z_1, z_2, z_3, \ldots, z_n, \ldots$$

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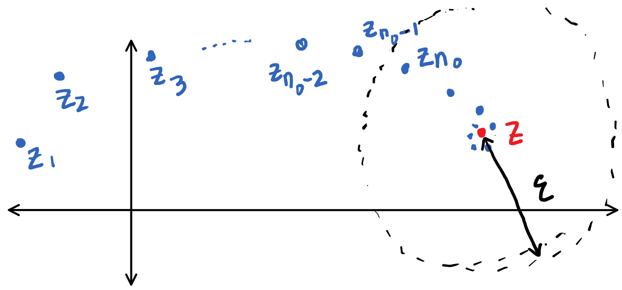
Definition (Limit of a Sequence) A sequence z_n has a limit $z \in \mathbb{C}$ if for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$n \geq n_0 \quad \text{implies} \quad |z_n - z| < \varepsilon.$$

A sequence that has a limit is convergent and we write

$$\lim_{n\to\infty} z_n = z.$$

A sequence with no limit is divergent.



$\boxed{\text{Proposition}}$

(1) The limit of a convergent sequence is unique.

(2) If $z_n = x_n + iy_n$ is a sequence, then

$$\lim_{n\to\infty} x_n + iy_n = x + iy \iff \lim_{n\to\infty} x_n = x \text{ and } \lim_{n\to\infty} y_n = y.$$

Proof.

(1) Assume $\lim_{n\to\infty} z_n = z_1$ and $\lim_{n\to\infty} z_n = z_2$. Let $\varepsilon > 0$. Choose $n_1, n_2 \in \mathbb{N}$ such that

$$n \geq n_1 \implies |z_n - z_1| < \frac{\varepsilon}{2}$$

Then $n \geq n_2 \implies |z_n - z_2| < \frac{\varepsilon}{2}$; $n \geq \max\{n_1, n_2\}$, then

$$|z_1 - z_2| \leq |z_n - z_1| + |z_n - z_2| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

(2) $(\Rightarrow)$ Assume $\lim_{n\to\infty} x_n + iy_n = x + iy$. Let $\varepsilon > 0$. Choose

$n_0 \in \mathbb{N}$ such that $n \geq n_0$ implies $|x_n + iy_n - (x+iy)| < \varepsilon$. Then $|x_n - x| \leq |x_n - x + i(y_n - y)| < \varepsilon$.

But then $|y_n - y| \leq |x_n - x + i(y_n - y)| < \varepsilon$.

$(\Leftarrow)$ Similar to the argument in (1).

$\blacksquare$

$\boxed{\text{Example}}$ We show that

$$\lim_{n\to\infty} -1 + i\,\frac{(-1)^n}{n^2} = -1.$$

By the theorem

$$\lim_{n\to\infty} -1 + i\,\frac{(-1)^n}{n^2} = \lim_{n\to\infty} -1 + i \lim_{n\to\infty} \frac{(-1)^n}{n^2} = -1 + 0 = -1.$$

By definition, let $\varepsilon > 0$. Choose $n_0 > \frac{1}{\sqrt{\varepsilon}}$. Then for an $n \geq n_0$,

$$\left| -1 + i\,\frac{(-1)^n}{n^2} - (-1) \right| = \left| i\,\frac{(-1)^n}{n^2} \right| = \frac{1}{n^2} < \varepsilon.$$

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$\boxed{\text{Definition (Series)}}$ A series of complex numbers is a symbol

$$\sum_{n=1}^{\infty} z_n = z_1 + z_2 + z_3 + \cdots + z_n + \cdots$$

associated to a sequence z_n of complex numbers. A series has an associated sequence of partial sums

$$S_N = \sum_{n=1}^{N} z_n = \underbrace{z_1 + z_2 + \cdots + z_N}_{\text{sum the first } N \text{ terms}}.$$

A series is convergent if S_N is convergent. In this case, we write

$$\sum_{n=1}^{\infty} z_n = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \sum_{n=1}^{N} z_n.$$

The limit $\lim_{N\to\infty} S_N$ is called the sum of the series. A series that does not converge is divergent.

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Proposition Suppose that $z_n = x_n + i y_n$ is a sequence. Then

$$\sum_{n=1}^{\infty} z_n = X + iY \iff \sum_{n=1}^{\infty} x_n = X \quad \text{and} \quad \sum_{n=1}^{\infty} y_n = Y.$$

Proof. This is just the proposition for sequences applies to the partial sums. $\square$

According to the proposition, we can write

$$\sum_{n=1}^{\infty} x_n + iy_n = \sum_{n=1}^{\infty} x_n + i \sum_{n=1}^{\infty} y_n$$

provided that the series on the left converges or the two on the right converge.

Several results from calculus have counterparts in complex analysis:

Proposition (Test for Divergence) If $\sum_{n=1}^{\infty} z_n$ converges, then

$$\lim_{n\to\infty} z_n = 0.$$

Proof. Write $z_n = x_n + i y_n$. Then by the proposition, the series $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ converge. But these are series of real numbers, so from calculus $\lim_{n\to\infty} x_n = 0 = \lim_{n\to\infty} y_n$. Hence,

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} x_n + i \lim_{n\to\infty} y_n = 0.$$ $\blacksquare$

$\boxed{\text{Corollary}}$ If $\sum_{n=1}^{\infty} z_n$ converges, then there exists $M > 0$ such that $|z_n| \leq M$ for all $n \in \mathbb{N}$. That is, the sequence z_n is bounded.

Proof. If $\sum_{n=1}^{\infty} z_n$ converges, then $\lim_{n\to\infty} z_n = 0$. Choose $n_0 \in \mathbb{N}$ such that $n \geq n_0$ implies $|z_n| < 1$. Then put

$$M = \max\{1, |z_1|, |z_2|, \dots, |z_{n_0-1}|\}.$$

Then $|z_n| \leq M$ for all $n \in \mathbb{N}$. ∎

$\boxed{\text{Definition (Absolute Convergence)}}$ A series $\sum_{n=1}^{\infty} z_n$ is absolutely convergent if the series $\sum_{n=1}^{\infty} |z_n|$ of real numbers converges.

$\boxed{\text{Corollary (Absolutely Convergent Series Converge)}}$ If $\sum_{n=1}^{\infty} z_n$ is absolutely convergent, then it is convergent.

Proof. By assumption, the series $\sum_{n=1}^{\infty} |z_n|$ converges. Notice that $|x_n| \leq |z_n|$ and $|y_n| \leq |z_n|$ for all $n \in \mathbb{N}$. By the comparison test from calculus, the series

$$\sum_{n=1}^{\infty} |x_n| \quad \text{and} \quad \sum_{n=1}^{\infty} |y_n|$$

converge. Hence, the series $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ are absolutely convergent and hence (by calculus) converge. By the proposition, we can conclude that $\sum_{n=1}^{\infty} z_n$ converges. ∎

$\boxed{\text{Definition (Remainder of a Convergent Series)}}$ Suppose $\sum_{n=1}^{\infty} z_n$ is a convergent series and S its sum. The N^{th} remainder of the series is the complex number

$$\rho_N = S - S_N = S - \sum_{n=1}^{N} z_n = \sum_{n=1}^{\infty} z_n - \sum_{n=1}^{N} z_n.$$

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The remainder provides a convenient way to prove that $\sum_{n=1}^{\infty} z_n = S$. Just notice that

$$|S_N - S| = |\rho_N - 0|$$

So $\sum_{n=1}^{\infty} z_n = S$ if and only if $\lim_{N\to\infty} \rho_N = 0$. We will frequently make use of this.

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Power Series

Definition (Power Series) A power series is a series of

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots$$

where a_n is a sequence, $z_0 \in \mathbb{C}$ fixed, and z is any complex number in a prescribed region in $\mathbb{C}$. The associated sum, partial sum, and remainder depend on z, and are denoted

$$S(z), \quad S_N(z), \text{ and } \rho_N(z)$$

respectively.

Example (Geometric Series) We show that the geometric series $\sum_{n=0}^{\infty} a z^n$ is convergent when $|z|<1$. In fact,

$$\sum_{n=0}^{\infty} a z^n = \frac{a}{1-z} \qquad (|z|<1).$$

We compute the remainder:

$$\rho_N(z) = S(z) - S_N(z) = \frac{a}{1-z} - \sum_{n=0}^{N-1} a z^n$$

$$\left(1 + w + w^2 + \cdots w^{n-1} = \frac{1-w^n}{1-w} \right) \quad = \frac{a}{1-z} - a\left(\frac{1-z^N}{1-z}\right)$$

$$= a\left(\frac{z^N}{1-z}\right).$$

Hence, $|\rho_N(z)| = |a| \dfrac{|z|^N}{|1-z|}$. But the sequence of real numbers $\dfrac{|a|\,|z|^N}{|1-z|}$ converges to 0 if $|z|<1$ and diverges otherwise. Hence

$$\lim_{N\to\infty} \rho_N(z) = \begin{cases} 0, & |z|<1 \\ \text{diverges otherwise.} \end{cases}$$

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$\boxed{\text{Theorem (Taylor's Theorem)}}$ Suppose that f is analytic on an open disk $D_R(z_0)$. Then at each $z \in D_R(z_0)$, $f(z)$ has a convergent power series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

with coefficients

$$a_n = \frac{f^{(n)}(z_0)}{n!}.$$

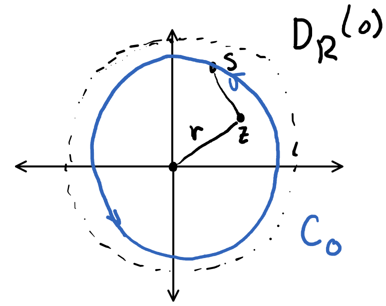
The series representation of f guaranteed by the theorem is called the Taylor Series of f about z_0.

Proof. First, assume that $z_0 = 0$ so that f is analytic $D_R(0)$.

Let $z \in D_R(0)$. Write $|z| = r$. Let r_0 be a real number such that $r < r_0 < R$. Let C_0 be the circle of radius r_0 centered at 0.

By Cauchy's Integral Formula, we have

$$f(z) = \frac{1}{2\pi i} \int_{C_0} \frac{f(s)}{s-z}\, ds .$$



Recall the formula :

$$1 + w + w^2 + \cdots w^{N-1} = \frac{1-w^N}{1-w} = \frac{1}{1-w} - \frac{w^N}{1-w} .$$

For any $N \in \mathbb{N}$, we can write

$$\frac{1}{s-z} = \frac{1}{s}\left(\frac{1}{1-\frac{z}{s}}\right) = \frac{1}{s}\left(\sum_{n=0}^{N-1}\left(\frac{z}{s}\right)^n + \frac{\left(\frac{z}{s}\right)^N}{1-\frac{z}{s}}\right)$$

$$= \sum_{n=0}^{N-1} \frac{z^n}{s^{n+1}} + \frac{z^N}{s^N(s-z)} .$$

We compute the remainder :

$$\rho_N(z) = f(z) - \sum_{n=0}^{N-1} \frac{f^{(n)}(0)}{n!} z^n$$

$$= \frac{1}{2\pi i}\int_{C_0} \frac{f(s)}{s-z}\, ds - \sum_{n=0}^{N-1} \frac{1}{\cancel{n!}} \frac{\cancel{n!}}{2\pi i} \int_{C_0} \frac{f(s)}{(s-0)^{n+1}} z^n\, ds$$

(Generalized Cauchy Integral Formula)

$$= \frac{1}{2\pi i}\int_{C_0} f(s)\left(\frac{1}{s-z} - \sum_{n=0}^{N-1} \frac{z^n}{(s-0)^{n+1}}\right) ds$$

$$= \frac{1}{2\pi i}\int_{C_0} f(s)\frac{z^N}{s^N(s-z)}\, ds$$

(by the formula for $\frac{1}{s-z}$).

Now, we prove that $\lim_{N\to\infty} \rho_N(z) = 0$. We have

$$|\rho_N(z)| = \frac{1}{2\pi}\left|\int_{C_0} f(s)\frac{z^N}{s^N(s-z)}\, ds\right|$$

$$\leq \frac{1}{2\pi} \max_{s\in C_0}\left|\frac{f(s)\, z^N}{s^N(s-z)}\right| \cdot 2\pi r_0$$

$$= \max_{s \in C_0} \frac{|f(s)|\,|z|^N}{|s|^N\,|s-z|} \cdot r_0$$

$|s-z| \geq ||s|-|z||$
$= |r_0 - r|$
$= r_0 - r$

$$\leq \frac{r^N \cdot r_0}{r_0^N (r_0 - r)} \max_{s \in C_0} |f(s)|$$

Just a constant

The sequence $\left(\frac{r}{r_0}\right)^N \cdot \frac{r_0}{r_0 - r} M$ converges to 0 since $\frac{r}{r_0} < 1$.

This proves $\lim_{N \to \infty} \rho_N(z) = 0$. This proves the claim when $z_0 = 0$.

Now assume that $z_0 \neq 0$ so that f is analytic on the disk $D_R(z_0)$. Then $f(z+z_0)$ is analytic on $D_R(0)$. By the preceding case, we can write

$$f(z + z_0) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} z^n$$

Now replace z with $z - z_0$ to get

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n.$$

This completes to proof. ∎

The Taylor series of f about $z_0 = 0$ is commonly referred to as a MacLauren series.

Examples (MacLauren Series of Elementary Functions) We will derive the following MacLauren Series representations of the most common elementary functions. We will frequently use these to compute MacLauren and Taylor series for other functions. You should memorize them!

(1) $\dfrac{1}{1-z} = \sum_{n=0}^{\infty} z^n$, $(|z|<1)$

(2) $e^z = \sum_{n=0}^{\infty} \dfrac{z^n}{n!}$, $(|z|<\infty)$

(3) $\sin z = \sum_{n=0}^{\infty} \dfrac{(-1)^n z^{2n+1}}{(2n+1)!}$, $(|z|<\infty)$

(4) $\cos z = \sum_{n=0}^{\infty} \dfrac{(-1)^n z^{2n}}{(2n)!}$, $(|z|<\infty)$

(5) $\sinh z = \sum_{n=0}^{\infty} \dfrac{z^{2n+1}}{(2n+1)!}$, $(|z|<\infty)$

(6) $\cosh z = \sum_{n=0}^{\infty} \dfrac{z^{2n}}{(2n)!}$, $(|z|<\infty)$

Solution.

(1) $\dfrac{1}{1-z} = \sum_{n=0}^{\infty} z^n$. Let $f(z) = \dfrac{1}{1-z}$. Then f has a singularity at $z=1$ so f is analytic on the disk $D_1(0)$. By Taylors theorem, f has a Maclaurin series on that disk. We have

$$f'(z) = \frac{d}{dz}(1-z)^{-1} = -1\,(1-z)^{-2}\cdot -1 = \frac{1}{(1-z)^2}$$

$$f''(z) = \frac{d}{dz}(1-z)^{-2} = -2\cdot(1-z)^{-3}\cdot -1 = \frac{2}{(1-z)^3}$$

$$\vdots$$

$$f^{(n)}(z) = \frac{n!}{(1-z)^{n+1}}$$

Hence, $f^{(n)}(0) = n!$. Hence

$$\frac{1}{1-z} = f(z) = \sum_{n=0}^{\infty} \frac{n!}{n!} z^n = \sum_{n=0}^{\infty} z^n.$$

(2) $e^z = \sum_{n=0}^{\infty} \dfrac{z^n}{n!}$. Since $f(z) = e^z$ is entire, it has a

MacLauren series everywhere, by Taylors theorem. We have

$$f^{(n)}(0) = e^0 = 1.$$

Hence, $$e^z = f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

(3) $\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$. We have

$$\begin{aligned}
\sin z &= \frac{1}{2i}\left(e^{iz} - e^{-iz}\right) \\
&= \frac{1}{2i}\left(\sum_{n=0}^{\infty} \frac{i^n z^n}{n!} - \sum_{n=0}^{\infty} \frac{(-i)^n z^n}{n!}\right) \\
&= \frac{1}{2i} \sum_{n=0}^{\infty} \frac{i^n z^n}{n!}\left(1 - (-1)^n\right) \\
&= \frac{1}{2i} \sum_{n=0}^{\infty} \frac{i^{2n+1} z^{2n+1}}{(2n+1)!} \cdot 2 \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}.
\end{aligned}$$

(4) $\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$. We can differentiate power series term by term. We have

$$\cos z = \frac{d}{dz}\sin z = \sum_{n=0}^{\infty} \frac{d}{dz} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}.$$

(5) $\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$. Recall $\sinh z = -i \sin iz$. Hence,

$$\sinh z = -i \sum_{n=0}^{\infty} \frac{(-1)^n (iz)^{2n+1}}{(2n+1)!}$$
$$= \sum_{n=0}^{\infty} (-1)^{n+1} \frac{i^{2n+2}}{(2n+1)!} z^{2n+1}$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(-1)^{n+1} z^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$$

(6) $\cosh z = \frac{d}{dz}\sinh z = \sum_{n=0}^{\infty} \frac{d}{dz} \frac{z^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$.

$\square$

Note: the power series in (2)-(6) are the usual Maclauren series when z is real. This provides additional justification that we chose the correct definitions when extending the elementary functions to the complex plane.

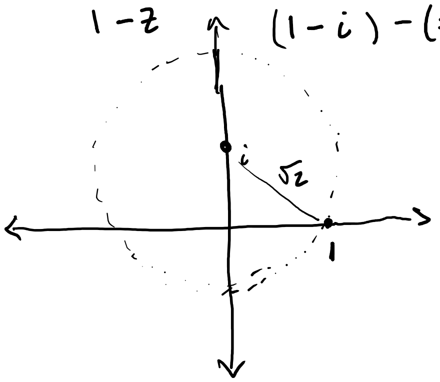
$\boxed{\text{Example}}$ We use the six MacLauren Series for the elementary functions to compute MacLauren Series or Taylor Series of other functions.

(a) MacLauren series of $\frac{1}{1+z}$. We have

$$\frac{1}{1+z} = \frac{1}{1-(-z)} \overset{(1)}{=} \sum_{n=0}^{\infty} (-z)^n = \sum_{n=0}^{\infty} (-1)^n z^n \quad \Big(|z| < 1 \Big)$$

(b) Taylor series for $\frac{1}{1-z}$ about $z_0 = i$. We have

$$\frac{1}{1-z} = \frac{1}{(1-i)-(z-i)} = \frac{1}{1-i} \cdot \frac{1}{1-\left(\frac{z-i}{1-i}\right)}$$



$$\overset{(1)}{=} \frac{1}{1-i} \sum_{n=0}^{\infty} \left(\frac{z-i}{1-i}\right)^n \quad \left(\begin{aligned} |z-i| < |1-i| \\ = \sqrt{2} \end{aligned} \right)$$

$$= \sum_{n=0}^{\infty} \frac{(z-i)^n}{(1-i)^{n+1}}.$$

(c) MacLauren series of $z^2 e^{2z}$. We have

$$z^2 e^{2z} \overset{(2)}{=} z^2 \sum_{n=0}^{\infty} \frac{(2z)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n z^{n+2}}{n!}$$

$$= \sum_{n=2}^{\infty} \frac{2^{n-2} z^n}{(n-2)!}. \quad //$$

Laurent Series

When f is not analytic at a point z_0, Taylors theorem cannot be applied. However, we can often find a series representation of f that involves negative powers of $z - z_0$.

Examples

(1) $f(z) = \dfrac{e^{-z}}{z^2}$. The function is not analytic at $z_0 = 0$ so we look for a series expansion involving powers of z. We have

$$\frac{e^{-z}}{z^2} \overset{(2)}{=} \frac{1}{z^2}\sum_{n=0}^{\infty} \frac{(-z)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{n-2}}{n!}$$

$$= \frac{1}{z^2} - \frac{1}{z} + \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{(n+2)!}$$

$$(0 < |z| < \infty)$$

(2) $f(z) = \dfrac{1+2z^2}{z^3+z^5}$.

$$\frac{1+2z^2}{z^3+z^5} = \frac{1}{z^3}\left(\frac{1+2z^2}{1+z^2}\right) = \frac{1}{z^3}\left(\frac{2(1+z^2) - 1}{1+z^2}\right)$$

$$= \frac{1}{z^3}\left(\frac{2(1+z^2)}{1+z^2} - \frac{1}{1+z^2}\right)$$

$$= \frac{1}{z^3}\left(2 - \frac{1}{1+z^2}\right)$$

$$\overset{(1)}{=} \frac{2}{z^3} - \frac{1}{z^3}\sum_{n=0}^{\infty} (-z^2)^n \qquad (0 < |z| < 1)$$

$$= \frac{2}{z^3} - \sum_{n=0}^{\infty} (-1)^n z^{2n-3}$$

$$= \frac{2}{z^3} - \frac{1}{z^3} + \frac{1}{z} - \sum_{n=2}^{\infty} (-1)^n z^{2n-3}$$

$$= \frac{1}{z^3} + \frac{1}{z} - \sum_{n=2}^{\infty} (-1)^n z^{2n-3}.$$

(3) $f(z) = \frac{e^z}{(z+1)^2}$. The singularity is at $z_0 = -1$ so we are looking for powers of $z+1$. We have

$$\frac{e^z}{(z+1)^2} = \frac{e^{z+1}}{e(z+1)^2} = \frac{1}{e(z+1)^2} \sum_{n=0}^{\infty} \frac{(z+1)^n}{n!}$$

$$= \frac{1}{e} \sum_{n=0}^{\infty} \frac{(z+1)^{n-2}}{n!} \qquad (0 < |z+1| < \infty)$$

$$= \frac{1}{e} \left[\frac{1}{(z+1)^2} + \frac{1}{(z+1)} + \sum_{n=0}^{\infty} \frac{(z+1)^n}{(n+2)!} \right] //$$

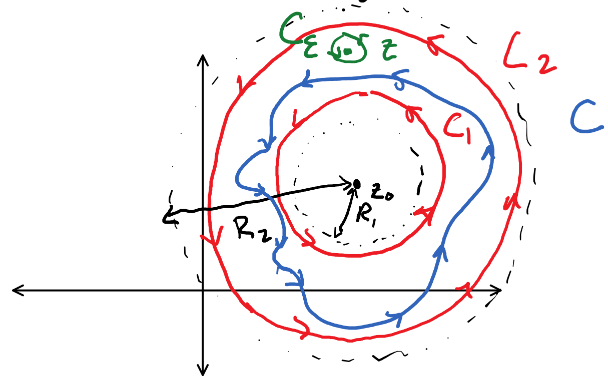
$\boxed{\text{Theorem (Laurent)}}$ Suppose that f is analytic on an annulus $R_1 < |z - z_0| < R_2$. Then f has a Laurent Series representation on that annulus

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n \frac{1}{(z-z_0)^n} \qquad (R_1 < |z - z_0| < R_2)$$

with coefficients given by

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} \, dz \quad \text{and} \quad b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{-n+1}} \, dz$$

where C is an positively oriented simple closed contour in the annulus whose interior contains z_0.



Proof. Initially, assume $z_0 = 0$. Let z be such that $R_1 < |z| < R_2$. Let C_1 and C_2 be circles (w/ positive orientation) with radii r_1 and r_2 such that

$$R_1 < r_1 < |z| < r_2 < R_2$$

and such that the contour C lies in between C_1 and C_2. Also let $\varepsilon > 0$ be so small that the circle $C_\varepsilon = C_\varepsilon(z)$ lies in between C_1 and C_2.

Now, we compute the remainder. First,

$$f(z) = \frac{1}{2\pi i} \int_{C_\varepsilon} \frac{f(s)}{s-z}\, ds \qquad \left(\begin{matrix}\text{Cauchy Int.}\\ \text{Formula}\end{matrix}\right)$$

$$= \frac{1}{2\pi i}\left(\int_{C_2} \frac{f(s)}{s-z}\, ds + \int_{C_1} \frac{f(s)}{z-s}\, ds \right) \qquad \left(\begin{matrix}\text{Cauchy}\\ \text{Goursat}\end{matrix}\right).$$

Recall, from proof of Taylor's Theorem:

$$\frac{1}{s-z} = \sum_{n=0}^{N-1} \frac{z^n}{s^{n+1}} + z^N \frac{1}{(s-z)s^N}$$

$$\frac{1}{z-s} = \sum_{n=0}^{N-1} \frac{s^n}{z^{n+1}} + s^N \frac{1}{(z-s)z^N}$$

$$= \sum_{n=1}^{N} \frac{s^{n-1}}{z^n} + s^N \frac{1}{(z-s)z^N}.$$

Then $$\rho_N(z) = f(z) - \sum_{n=0}^{N-1} a_n (z-0)^n + \sum_{n=1}^{N} b_n \frac{1}{(z-0)^n}$$

$$= \frac{1}{2\pi i}\left(\int_{C_2} f(s)\left(\frac{1}{s-z} - \sum_{n=0}^{N-1} \frac{z^n}{s^{n+1}}\right) ds \right.$$

$$\left. + \int_{C_1} f(s)\left(\frac{1}{z-s} - \sum_{n=1}^{N} \frac{z^{-n}}{s^{-n+1}}\right) ds \right)$$

$$= \frac{1}{2\pi i}\left(\int_{C_2} \frac{f(s)\, z^N}{(s-z)\, s^N} + \int_C \frac{f(s)\, s^N}{(z-s)\, z^N} \right)$$

Then
$$|\rho_N(z)| \le \frac{1}{2\pi}\left| \int_{C_2} \frac{f(s) z^N}{(s-z) s^N}\, ds \right| + \frac{1}{2\pi}\left| \int_C \frac{f(s) s^N}{(z-s) z^N}\, ds \right|$$

You can show that both integrals on the right converge to 0 as $N \to \infty$ using the T.I. for contour integrals, as in the proof of Taylors theorem. This proves the claim when $z_0 = 0$.

Suppose $z_0 \neq 0$ and assume f satisfies the conditions of the theorem. Define $g(z) = f(z+z_0)$. Since f is analytic on $R_1 < |z-z_0| < R_2$, g is analytic on $R_1 < |z| < R_2$. By the case we just proved,

$$g(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n \frac{1}{z^n}$$

with
$$a_n = \frac{1}{2\pi i} \int_\Gamma \frac{g(z)}{z^{n+1}}\, dz \qquad b_n = \frac{1}{2\pi i} \int_\Gamma \frac{g(z)}{z^{-n+1}}\, dz$$

where Γ is the contour obtained from C by translation by z_0. To finish the proof, replace g by $f(z+z_0)$ and replace z by $z - z_0$. This completes the proof. ∎

Example Laurent series are rarely found by using the integral expressions. Usually they are found by making use of the 6 MacLauren series for elementary functions.

(1) $f(z) = \dfrac{1}{z(1+z^2)}$. The singularities are at $0, i, -i$,

so the function is analytic on $0 < |z| < 1$. By Laurent's theorem, f has a Laurent series on this annulus. We have

$$\frac{1}{z(1+z^2)} = \frac{1}{z}\left(\frac{1}{1+z^2}\right) = \frac{1}{z}\sum_{n=0}^{\infty}(-z^2)^n = \sum_{n=0}^{\infty}(-1)^n z^{2n-1}$$

$$= \frac{1}{z} + \sum_{n=0}^{\infty}(-1)^{n+1} z^{2n+1}.$$

Fun fact: $b_1 = 1$ so by the Laurent theorem

$$2\pi i = 2\pi i\, b_1 = \int_C f(z)\,dz = \int_C \frac{1}{z(1+z^2)}\,dz$$

where C is any positively oriented simple closed contour about 0 in the annulus.

(2) $f(z) = e^{1/z}$. The singularity is at $z_0 = 0$. The function is analytic on $0 < |z| < \infty$. We have

$$e^{\frac{1}{z}} = \sum_{n=0}^{\infty}\frac{\left(\frac{1}{z}\right)^n}{n!} = \sum_{n=0}^{\infty}\frac{1}{z^n n!} = 1 + \frac{1}{z} + \frac{1}{z^2 \cdot 2!} + \cdots.$$

Notice that $b_1 = 1$ so by Laurent's theorem

$$2\pi i = 2\pi i\, b_1 = \int_C e^{1/z}\,dz \quad \text{where}$$

C is any simple closed contour about 0.

(3) $f(z) = \frac{z+1}{z-1}$ has a singularity at $z=1$. We can find a Taylor series on the disk $|z|<1$ and a Laurent series on the annulus $1<|z|<\infty$.

On $|z|<1$: $\frac{z+1}{z-1} = -(z+1)\frac{1}{1-z} = -(z+1)\sum_{n=0}^{\infty} z^n$

$$= -\sum_{n=0}^{\infty} z^{n+1} - \sum_{n=0}^{\infty} z^n$$

$$= -1 - 2\sum_{n=0}^{\infty} z^{n+1}$$

$$= -1 - 2\sum_{n=1}^{\infty} z^n .$$

On $1<|z|<\infty$: this condition implies $|\frac{1}{z}|<1$. We have

$$\frac{z+1}{z-1} = \frac{z}{z} \cdot \frac{1+\frac{1}{z}}{1-\frac{1}{z}} = \frac{1}{1-\frac{1}{z}} + \frac{1}{z}\left(\frac{1}{1-1/z}\right)$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^n} + \frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n}$$

$$= 1 + 2\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}$$

$$= 1 + 2\sum_{n=1}^{\infty}\frac{1}{z^n} .$$

(4) $f(z) = \frac{1}{(z-z_0)^{n+1}}$, $n \geq 0$. This is analytic on the annulus $0<|z-z_0|<\infty$. In fact, $f(z)$ is already a Laurent series. We will compute

$$\frac{1}{2\pi i}\int_C \frac{1}{(z-z_0)^{(n+1)-m}}\, dz$$

for any $m \geq 0$ where C is any simple closed contour about z_0. By Laurents theorem

$$b_{m+1} = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{-(m+1)+1}}\, dz$$

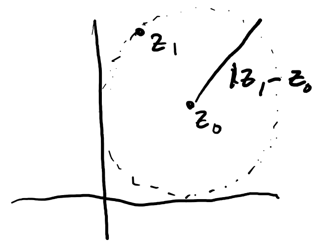
$$= \frac{1}{2\pi i} \int \frac{1}{(z-z_0)^{(n+1)-m}}\, dz$$

But $b_{m+1} = \begin{cases} 1, & m=n \\ 0, & \text{otherwise.} \end{cases}$

//

Absolute & Uniform Convergence

Theorem (Power Series Converge Absolutely) If a power series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ converges when $z = z_1$, then it converges absolutely on the disk $D_R(z_0)$, $R = |z_1 - z_0|$.



Proof. Assume the series converges at z_1. Then the sequence $a_n(z-z_0)^n$ is bounded. Choose $M>0$ so that $|a_n(z-z_0)^n| \leq M$ for all $n \in \mathbb{N}$. Now let $z \in D_R(z_0)$ so that $|z-z_0| < R = |z_1 - z_0|$. Write

$$\rho = \frac{|z-z_0|}{|z_1 - z_0|}.$$

Then $$|a_n(z-z_0)^n| = |a_n(z_1-z_0)^n| \left| \frac{(z-z_0)}{(z_1-z_0)} \right|^n$$

$$\leq M \rho^n.$$

But the series $\sum_{n=0}^{\infty} M\rho^n$ is a convergent geometric series since $\rho < 1$. Hence, by the comparison test, the series $\sum_{n=0}^{\infty} |a_n (z-z_0)^n|$ converges. $\blacksquare$

The theorem asserts that if a series converges at a point $z_1 \neq z_0$, then it converges on a disk $D_{|z_1 - z_0|}(z_0)$. The largest disk with this property is called the disk of convergence or circle of convergence. According to the theorem, a series does not converge at any point outside its disk of convergence.

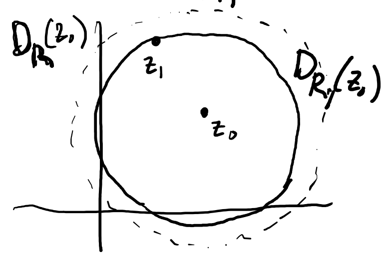
Definition (Uniform Convergence of Series) Consider a power series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ with disk of convergence $D_R(z_0)$.

Let S be a region in the disk. We say that the series converges uniformly on S if for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$n \geq n_0 \text{ and } z \in S \text{ implies } |\rho_n(z)| < \varepsilon.$$

In other words, $n_0 = n_0(\varepsilon)$ depends only on ε and not on z.

Theorem (Uniform Convergence of Series) If z_1 is a point inside the disk of convergence $D_R(z_0)$ of a series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$, then the series converges uniformly on the closed disk $\overline{D_{R_1}(z_0)}$ where $|z_1 - z_0| = R_1$.



Proof. By the preceding theorem, the series

$$\sum_{n=0}^{\infty} |a_n (z_1 - z_0)^n|$$

converges. Write the remainders of each series:

$$\rho_N(z) = \lim_{m\to\infty} \sum_{n=N}^{m} a_n (z-z_0)^n$$

$$\sigma_N = \lim_{m\to\infty} \sum_{n=N}^{m} \left| a_n (z_1 - z_0)^n \right| ,$$

Consider $z \in \overline{D_{|z_1-z_0|}(z_0)}$. Then $|z-z_0| \le |z_1-z_0|$. Hence,

$$|\rho_N(z)| = \lim_{m\to\infty} \left| \sum_{n=N}^{m} a_n (z-z_0)^n \right|$$

$$\le \lim_{m\to\infty} \sum_{n=N}^{m} |a_n| \, |z-z_0|^n$$

$$\le \lim_{m\to\infty} \sum_{n=N}^{m} |a_n| \, |z_1-z_0|^n = \sigma_N .$$

Let $\varepsilon > 0$. Choose $n_0(\varepsilon) \in \mathbb{N}$ such that $N \ge n_0(\varepsilon)$ implies $|\sigma_N| < \varepsilon$. Hence, $N \ge n_0(\varepsilon)$ and $z \in \overline{D_{|z_1-z_0|}}(z_0)$ implies $|\rho_N(z)| \le \sigma_N < \varepsilon$.

∎

$\boxed{\text{Theorem (Continuity of Power Series)}}$ A power series

$$S(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

is a continuous function on its disk of convergence.

Proof. Let $D_R(z_0)$ be the disk of convergence and let $z_1 \in D_R(z_0)$. Let $\varepsilon > 0$. Since the power series converges uniformly, choose $N(\varepsilon) \in \mathbb{N}$ such that for all $z \in \overline{D_{|z_1-z_0|}(z_0)}$,

$$N \geq N(\varepsilon) \implies |\rho_N(z)| < \frac{\varepsilon}{3}.$$

Also, since $S_N(z)$ is a polynomial for each $N \in \mathbb{N}$, it is a continuous function. Fix $N_0 = N(\varepsilon)+1$. Choose $\delta > 0$ such that

$$|z - z_1| < \delta \implies |S_{N_0}(z) - S_{N_0}(z_1)| < \frac{\varepsilon}{3}.$$

Then $|z - z_1| < \delta$ implies

$$\begin{aligned} |S(z) - S(z_1)| &= |S_{N_0}(z) + \rho_{N_0}(z) - (S_{N_0}(z_1) + \rho_{N_0}(z_1))| \\ &\leq |S_{N_0}(z) - S_{N_0}(z_1)| + |\rho_{N_0}(z)| + |\rho_{N_0}(z_1)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

∎

P2(b), P6

Theorem (Integrating Power Series) Let C be any contour interior to the disk of convergence of the power series

$$S(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

Let $g(z)$ be any function continuous on C. Then

$$\int_C g(z) S(z)\, dz = \sum_{n=0}^{\infty} a_n \int_C g(z)(z - z_0)^n\, dz.$$

Proof. Denote by $\sigma_N(z)$ the remainder of the series on the right. Write $\rho_N(z) = S(z) - S_N(z)$. We have

$$\begin{aligned} \sigma_N(z) &= \int_C g(z) S(z)\, dz - \sum_{n=0}^{N-1} a_n \int_C g(z)(z - z_0)^n\, dz \\ &= \int_C g(z)\left(S(z) - \sum_{n=0}^{N-1} a_n (z - z_0)^n\right) dz \end{aligned}$$

$$= \int_C g(z)\, \rho_N(z)\, dz.$$

Let $\varepsilon > 0$. Since g is continuous on C, choose $M > 0$ such that

$$|g(z)| \leq M \quad \text{for all } z \in C.$$

Since $S(z)$ is uniformly convergent on its disk of convergence $D_R(z_0)$, choose $N(\varepsilon) \in \mathbb{N}$ such that for all $z \in D_R(z_0)$,

$$N \geq N(\varepsilon) \implies |\rho_N(z)| < \frac{\varepsilon}{M \cdot \text{length}(C)}.$$

Then by the Triangle Ineq. for contour integrals,

$$|\sigma_N(z)| = \left| \int_C g(z)\, \rho_N(z)\, dz \right|$$

$$\leq \max_{z \in C} |g(z)| \cdot |\rho_N(z)| \cdot \text{length}(C)$$

$$\leq M \cdot \frac{\varepsilon}{M\,\text{length}} \cdot \text{length}(C) = \varepsilon.$$

This proves that $\lim_{N \to \infty} \sigma_N(z) = 0$.

$\blacksquare$

$\boxed{\text{Corollary (Power Series are Analytic)}}$ A power series

$$S(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

is an analytic function on its disk of convergence.

Proof. By one of the theorems, $S(z)$ is continuous on its disk of convergence. Let C be any closed contour lying inside the disk of convergence. Then

$$\int_C S(z)\,dz = \sum_{n=0}^{\infty} a_n \int_C (z-z_0)^n\,dz \qquad \left(\text{by the theorem}\right)$$

$$= \sum_{n=0}^{\infty} a_n \cdot 0 \qquad \left(\text{since } (z-z_0)^n \text{ has an anti-derivative}\right)$$

$$= 0.$$

By Morera's theorem, $S(z)$ is an analytic function! ■

P4

$\boxed{\text{Example}}$ The function

$$f(z) = \begin{cases} \dfrac{\sin z}{z} & , z \neq 0 \\ 1 & , z = 0 \end{cases}$$

is entire. For any $z \in \mathbb{C}$, we can write

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}.$$

When $z \neq 0$, we have

$$\frac{\sin z}{z} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n+1)!}.$$

But when $z=0$,

$$1 = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n+1)!}$$

Hence, $f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n+1)!}$ for any $z \in \mathbb{C}$. //

$\boxed{\text{Theorem (Differentiating Power Series)}}$ A power series

$$S(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

can be differentiated term-by-term. Precisely, at each point interior to the disk of convergence,

$$S'(z) = \sum_{n=1}^{\infty} n \cdot a_n (z - z_0)^{n-1}.$$

Proof. Let $D_R(z_0)$ be the disk of convergence and let $z \in D_R(z_0)$. Let C be a simple closed positively oriented contour interior to $D_R(z_0)$ and surrounding z. Then

$$\begin{aligned}
S'(z) &= \frac{1}{2\pi i} \int_C \frac{S(w)}{(w-z)^2}\, dw && \left(\begin{array}{c}\text{Cauchy's Int.}\\ \text{formula}\end{array}\right) \\
&= \int_C g(w)\, S(w)\, dw && \left(g(w) = \frac{1}{2\pi i} \cdot \frac{1}{(w-z)^2} \right) \\
&= \sum_{n=0}^{\infty} a_n \int_C g(w)(w-z_0)^n\, dz && \left(\begin{array}{c}\text{Integrating}\\ \text{Power}\\ \text{series}\end{array}\right) \\
&= \sum_{n=0}^{\infty} a_n \cdot \frac{1}{2\pi i} \int_C \frac{(w-z_0)^n}{(w-z)^2}\, dz \\
&= \sum_{n=0}^{\infty} a_n \frac{d}{dw}(w-z_0)^n \Big|_{w=z} && \left(\begin{array}{c}\text{Cauchy}\\ \text{Integral}\\ \text{Formula}\end{array}\right) \\
&= \sum_{n=0}^{\infty} a_n\, n\, (z-z_0)^{n-1}.
\end{aligned}$$

■

Theorem (Uniqueness of Taylor Series) If a power series
$$\sum_{n=0}^{\infty} a_n (z-z_0)^n$$
converges to a function $f(z)$ on a disk $D_R(z_0)$, then it is the Taylor series of f about z_0.

Proof. We need to show that $a_n = \frac{f^{(n)}(z_0)}{n!}$. Consider $g(z) = \frac{1}{2\pi i} \cdot \frac{1}{(z-z_0)^{n+1}}$ where $n \geq 0$. Let C be a circle centered at z_0 w/ radius $r < R$.

$$\begin{aligned}
\frac{f^{(n)}(z_0)}{n!} &= \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz \qquad \left(\text{Cauchy Int. Formula}\right) \\
&= \int_C g(z) \sum_{m=0}^{\infty} a_m (z-z_0)^m \, dz \\
&= \sum_{m=0}^{\infty} a_m \int_C g(z)\,(z-z_0)^m \, dz \qquad \left(\text{Integrating Power Series}\right) \\
&= \sum_{m=0}^{\infty} \frac{a_m}{2\pi i} \int_C \frac{1}{(z-z_0)^{(n+1)-m}}\, dz \\
&= a_n \qquad \left(\text{by an example}\right).
\end{aligned}$$

$\blacksquare$

Theorem (Uniqueness of Laurent Series) If a series
$$\sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n \frac{1}{(z-z_0)^n}$$
converges to a function $f(z)$ on an annulus $R_1 < |z-z_0| < R_2$, then it is the Laurent series for f on that annulus.

Proof. Similar to the proof of uniquess of Taylor series. $\square$

Multiplication of Power Series

Suppose two power series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} b_n (z-z_0)^n$$

converge on a disk $D_R(z_0)$. Then f and g are analytic on that disk and hence so is $f \cdot g$, by the product rule. Hence, $f \cdot g$ has a Taylor series on $D_R(z_0)$:

$$f(z)\, g(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n$$

with coefficients

$$\frac{n!}{k!(n-k)!}$$

$$\begin{aligned} c_n &= \frac{(fg)^{(n)}(z_0)}{n!} \\ &= \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(z_0)\, g^{(n-k)}(z_0) \qquad \text{(Liebniz)} \\ &= \sum_{k=0}^{n} \frac{f^{(k)}(z_0)}{k!} \frac{g^{(n-k)}(z_0)}{(n-k)!} \\ &= \sum_{k=0}^{n} a_k b_{n-k}. \end{aligned}$$

Usually, only the first several terms are required.

They can be found by formally multiplying the series like polynomials.

$\boxed{\text{Example}}$ Find the MacLaurin series for

$$f(z) = \frac{\sinh z}{1+z}.$$

The function $\sinh z$ and $\frac{1}{1+z}$ are analytic on the unit disk. We have

$$\sinh z \cdot \frac{1}{1+z} = \left(\sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}\right)\left(\sum_{n=0}^{\infty} (-1)^n z^n\right)$$

$$= \left(z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right)\left(1 - z + z^2 - \cdots\right)$$

$$\begin{aligned} = \; & z + \frac{z^3}{3!} + \frac{z^5}{5!} \\ & - z^2 - \frac{z^4}{3!} - \frac{z^6}{5!} \\ & + z^3 + \frac{z^5}{3!} + \frac{z^7}{5!} \\ & - z^4 - \frac{z^6}{3!} - \frac{z^8}{5!} + \end{aligned}$$

$$= z - z^2 + \frac{7}{6} z^3 - \frac{7}{6} z^4 + \cdots$$

//

Similarly, if $f(z)$ and $g(z)$ are analytic on a disk $D_R(z_0)$ and $g(z) \neq 0$ on $D_R(z_0)$, then we can write

$$\frac{f(z)}{g(z)} = \sum_{n=0}^{\infty} d_n (z - z_0)^n$$

where
$$d_n = \frac{\left(\frac{f}{g}\right)^{(n)}(z_0)}{n!}.$$

In fact, the coefficients turn out to be the same as those found by dividing the series like polynomials.

Example Find the Laurent series for
$$f(z) = \frac{1}{\sinh z}$$
on the annulus $0 < |z| < \pi$. We have
$$\frac{1}{\sinh z} = \frac{1}{\sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}} = \frac{1}{z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots}$$
$$= \frac{1}{z}\left(\frac{1}{1 + \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots}\right)$$

The series $1 + \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots$ is nonzero on the disk $|z| < \pi$, so we can divide.

$$\begin{array}{r|l}
 & 1 - \frac{z^2}{3!} + \left(\frac{1}{(3!)^2} - \frac{1}{5!}\right) z^4 + \cdots \\ \hline
1 + \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots & 1 \\
 & -\left(1 + \frac{z^2}{3!} + \frac{z^4}{5!} + \right) \\ \hline
 & -\frac{z^2}{3!} - \frac{z^4}{5!} - \cdots \\
 & -\left(-\frac{z^2}{3!} - \frac{z^4}{(3!)^2} - \quad\right) \\ \hline
 & \left(\frac{1}{(3!)^2} - \frac{1}{5!}\right) z^4 + \cdots
\end{array}$$

Hence,

$$\frac{1}{\sinh z} = \frac{1}{z}\left(1 - \frac{z^2}{3!} + \left(\frac{1}{(3!)} - \frac{1}{5!}\right)z^4 + \dots\right)$$

$$= \frac{1}{z} - \frac{z}{6} + \frac{7}{360}z^3 - \dots$$

//